Continuum, Chaos and Metronomes - A Fractal Friendship

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In the Beginning Was Chaos

When I met Ligeti in 1985, I had no idea what impact he would have on the rest of my life. My friend, the physicist Peter Richter, and I had just completed a major exhibition for the Goethe-Institut ("Beauty in Chaos – Images from the Theory of Complex Systems"; "Frontiers of Chaos – Computer Graphics Face Complex Dynamics"), which displayed images from mathematics never before seen. Two identical copies of the exhibition were made, and they began their world tour almost simultaneously at the Museum of Modern Art in Oxford and the San Francisco Exploratorium. The images showed mathematical computer experiments from Chaos Theory and Fractal Geometry. Ligeti had heard about some of the images through our mentor, Nobel laureate Manfred Eigen, and was electrified. He urgently wanted to know what the images and experiments were all about. This led to an encounter between two people who previously knew nothing of each other, and for both, entirely new worlds opened up. I myself had always been very fond of music, but unfortunately had the widespread problems with New Music. Therefore, Ligeti was not a familiar name to me. Today, I can only say, how fortunate that was. Because I approached him completely naively and without bias.

With our acquaintance, which soon turned into a heartfelt friendship, everything changed. New Music, and especially Ligeti's work, became a fixed point in my life. Anyone who had the pleasure of exchanging ideas with Ligeti will find my descriptions bland and pale, because his manner, his keen intellect, his probing style of questioning, and his infinitely softly packaged yet stinging criticism can hardly be conveyed in words. As different as the origins and work of Ligeti, the musician, and me, the natural scientist, may be, we shared the observation that the other was doing and knowing something that had remarkable points of contact and interconnections with one's own work. And that in uncovering these relationships, we felt a singular enrichment and fostered an increase in knowledge. He paved the way for me into new music, and I opened windows for him into Chaos Theory, Fractal Geometry, and many other mathematical fields. I remember well a long conversation in the Parkhotel Bremen after a concert of his works on the occasion of the Musikfest Bremen in 1997. The concert was a great success. As the anchor piece, the Deutsche Kammerphilharmonie Bremen with Christian Tetzlaff played the Violin Concerto. The concert took place in a very unusual venue: the space flight hall of the DASA (Daimler Benz Aerospace Airbus). Usually, spacecraft objects are assembled there. Ligeti and I were scheduled to guide the audience through the concert with segments of conversation. The concert was sold out, certainly also because of the announced presence of the master. I had prepared well because I knew that Ligeti would hardly stick to a pre-arranged topic limitation. His associative nature often led him from one subject to very distant observations. And that was precisely the charm of the conversations. On stage in front of 1000 spectators, however, this is a challenge for the moderator that can only be mastered with careful preparation. And that preparation was about to pay off, because when Ligeti arrived in the hall in the late afternoon, coming from Hamburg, he said without hesitation: “Impossible! That won't work. My concerto cannot be played here. The acoustics are miserable.” Now Christian Tetzlaff was in trouble. He had prepared excellently and adjusted to the difficult acoustics with the conductor Christian Hommel and, of course, could not contradict the master. Without the anchor piece, the conversation with Ligeti during the concert also made less sense. There was a back and forth. I tried to throw the weight of our friendship into the balance to save the situation. Finally, Ligeti offered the compromise to play the concert but not in his presence. He went to the beautiful Parkhotel, and I had 30 minutes to prepare for a solo role as moderator. The audience accepted that Ligeti was unwell, everything went well, and the Violin Concerto was celebrated. That was him. Ligeti tolerated no compromises on quality. The next day, the two friends met at the Parkhotel, and the conversation went on for hours about the role of infinity and its manifestations in mathematics, right up to Gödel's incompleteness theorem. One can chew on that for a few hours, and he did. I remember very well an outburst of pure joy from Ligeti when I explained to him how Bernhard Riemann placed infinity into a single point – the North Pole of a number sphere – and thus made infinity practically manageable. Of course, on this occasion, he wanted to know, as always: “What's new in Chaos Theory?” Incidentally, Ligeti didn't like the term Chaos at all. He found it too populistic and overestimated my possibilities of convincing the mathematical community to choose a better term.

Chaos Theory had its peak in public perception in the mid-eighties. In retrospect, one can certainly speak of a hype. And scientific hypes usually have two sides. First, the expectations are incredibly high, and then follows a disappointment about the unfulfilled expectations. In fact, the discovery of chaos has sustainably expanded and changed our mathematical-scientific worldview, and quietly, countless applications have emerged in the meantime. What is chaos and what is it about?

Lessons from the Weather: Chaos

On April 16, 2008, Edward N. Lorenz died at the age of 90. As one of the fathers of Chaos Theory, he enriched the sciences of the 20th century with a third groundbreaking insight, alongside Quantum Theory and the Theory of Relativity.

People have always pursued weather forecasts with great ambition. Until the beginning of the last century, they had a simple strategy: meteorologists had built up a large treasure trove of observed weather patterns in their temporal sequence and based a prognosis on how an current weather pattern best fit a previous sequence. That seemed plausible, because if the weather pattern today is like it was, say, on a certain day two years ago, then in the future it should develop as it did then. However, this method was not crowned with great success.

In 1922, the great British meteorologist Lewis Fry Richardson presented a revolutionary new approach. He created a precise mathematical model – the formulas and equations fill an entire book, by the way – that describes the physical development of weather patterns and proposes to calculate the weather of the future from the measurements of the weather of the present. With this gigantic individual achievement, he integrates the hitherto only observational and interpretive archaic science of meteorology into the modern natural sciences. But the dream of practically using Richardson's work is only fulfilled after the first computers are developed at the end of the forties. Since then, meteorologists have struggled to achieve Richardson's vision with increasingly powerful computers, and it is initially the undisputed conviction of *all* meteorologists that the goal will be achieved.

It was in this spirit that Edward N. Lorenz was working when he discovered something peculiar in 1960, which he later called the “Butterfly Effect.” Although he initially only considered a mathematically simplified weather model, he found that the slightest uncertainties in the measurements describing today's weather can lead to the consequence that completely different weather patterns can emerge in the long term! A few years later, the term "Chaos" was coined for such situations. *Chaos is the practical loss of predictability in the presence of a strictly valid, mathematically available natural law.* Practically here means that the prediction is made based on a measurement, and measurements always have a certain uncertainty. Lorenz's discovery relativizes Richardson's dream. Today we know that the weather can be predicted for two to three weeks at best. An example of how *a* lonely scientist can shake the conviction of an entire scientific community. The significance of Lorenz also lies in the fact that he pioneered revolutionary mathematical paths in his methods, thus creating the first components of Chaos Theory, which subsequently proved so fruitful that natural scientists found chaos almost everywhere in nature in the seventies and eighties, and today one rightly speaks of a new image of nature. Modern, mathematically conceived natural science had nourished the dream of being able to see into the future since Galileo and Newton. Chaos Theory says that reliable prognoses for developments in nature are rather the exception. The future is determined by the present, but in many cases, we practically cannot recognize it. Or in other words, the future is much more open than we can imagine.

The history of Chaos Theory also shows that majority opinions of scientists do not guarantee greater certainty with regard to the claim to truth, and that mathematics makes our limits in recognizing or even influencing the future unequivocally clear to us.

Finding chaos in the mathematical-scientific sense means metaphorically finding butterflies in a matter, i.e., discovering in a process, in a development, in an event, how the smallest uncertainties or errors or disturbances can cause powerful, surprising effects. I myself am currently working in my research on improving medical diagnostics and therapy and about 10 years ago discovered a trace of chaos in the assessment of risks in liver surgery.

Surgical Risk Analysis

To understand the problem of surgical risk analysis during the resection – that is: the surgical removal – of a tumor from an organ, it is important to be aware of the following: There is no sure diagnostic method that tells us exactly where the tumor ends and where it begins. This is impossible according to the current state of medical knowledge. Therefore, the surgeon proceeds by trying to operate in what is certainly healthy tissue, which means he plans a safety margin around the tumor he sees on computed tomography slices, which is then hopefully such that after the operation the pathologist actually finds the tumor enclosed within this safety margin or at least finds no tumor cells in the resection margin. Because if some tumor cells remain in the organ, the operation was practically meaningless. The problem now, however, is that the organ itself has all sorts of structures; it is networked locally and also over distance. If one includes such a safety margin when operating on the tumor, this networking structure is inevitably injured. As a result, there will then be organ failures somewhere in the distance. How do these risks depend on the tumor's location? In sophisticated experiments, I have found that sometimes the smallest differences in the position of a tumor can have large differences in the effect on surgical risks – so, butterflies in the risk distribution in tumor surgery.

Chaos and Crash Tests

Our need for safety demands car bodies that protect the occupants in an accident. The automotive industry performs crash tests for this purpose. In the past, a car had to be sacrificed for every test. Today, these tests are carried out in computers. An incredible discovery was made during the development of the mathematical-engineering procedures that make these simulated tests possible and accurately reproduce reality. Butterflies were found. That is, sometimes the crash results differ considerably if, for example, the car impacts from a direction deviating by a few degrees. So, small differences can unfold a great effect. And mathematical methods have been used to precisely determine where the butterflies are constructively located. Or, which properties of components are responsible for the butterflies and how they can be avoided.

Chaos and Ligeti and Super Signals

Of all Ligeti's works, two stand out for me because of their experimental, scientific character. This is evident in *Poème Symphonique* for 100 metronomes (1962), and the setup for the piece and the course of the performance are already reminiscent of an experiment. In *Continuum* (1968) for harpsichord, merely by listening to the piece, one would hardly get the idea that one is also witnessing a cleverly conceived experiment, because the composition has a wonderful musicality and tension that completely captivates one. One might think of a revival of the solo cadenza in the 1st movement of the 5th Brandenburg Concerto or the C minor Prelude from the 1st volume of the Well-Tempered Clavier and experience Bach completely anew. But the thought that Ligeti is conducting experiments in perception psychology with us does not cross our minds. Yet, when we read the instruction in the score:

> Prestissimo = extremely fast, so that the individual notes are barely
> perceptible, but merge into a continuum.
> Play very evenly, without any articulation. The correct tempo
> has been achieved when the piece (without the final phase) takes less than
> 4 minutes. The vertical dotted lines are not bar lines
> (there is no beat or meter here), but serve only for orientation.
> B. Schott's Söhne, Mainz, 1970

it immediately becomes clear that this is not just about a usual speed indication, but about the realization of an effect. It must be played at the physiological limit, or actually beyond it, so that the struck notes merge into a continuum. An effect that we use in a film projector to turn single frames, which are shown staccato at approx. 25 pictures per second, into a continuous flow. Yes, indeed, the film projector shows still single images, but so fast that our perception can no longer keep up with seeing the images as still single images and embeds them in a flow. The flowing movement of the film is thus an addition within us. It is not shown. One could speak of a virtual movement. Tones are actually something similar. If I strike a table with a stick in absolute regularity, say 60 times a minute, then I hear stick beats. If I could now strike something, which is, however, beyond our physiological possibilities, namely 440 times per second, then I would hear the concert pitch A. The beats merge into a tone. If two people strike now, what happens then? Let's say one 440 times per second and the other 441 times per second. What do we hear? A "mixture" of two tones perhaps? No, we hear a rhythm that pulsates once per second. We can experience something related in our visual perception as well. Figure 1 shows

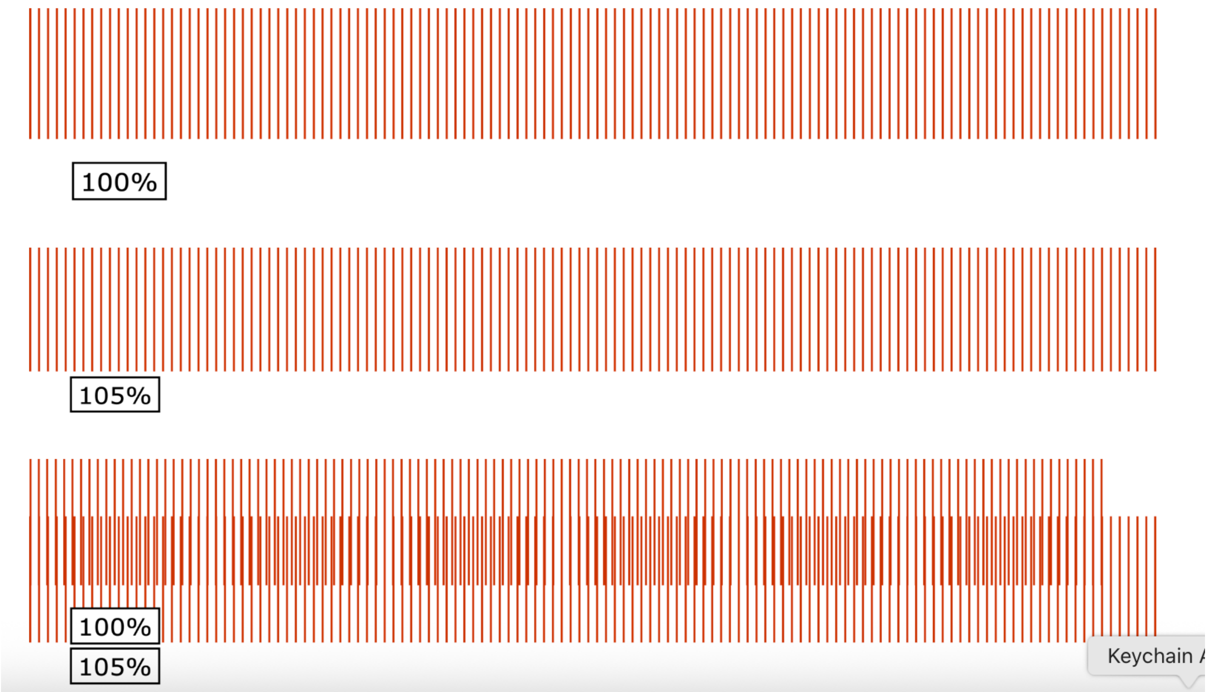

Figure 1.

a simple experiment. Above is a sequence of bars at the same (small) distance, visualizing a simple rhythm. Below that are bars again at the same distance, but at a slightly greater distance (slower rhythm) than above; the distances are extended by exactly 5%. And finally, we see the two upper bar sequences partially interleaved below and observe that where the two intermingle, something like a new, very slow rhythm emerges (here slowed down by about a factor of 20). One could say, when two closely adjacent rhythms rub against each other, new rhythms emerge.

These facts seem completely banal to me today, and as a natural scientist, I should have known about them since my student days at the latest. But the truth is that I owe my awareness of these connections to the conversations with Ligeti, and since then, I see and hear differently, newly and enriched. In our conversations, the names Steve Reich and Conlon Nancarrow often came up. Initially, I simply accepted these names and did not pursue them further. But Ligeti's repeated reference finally helped me over the fence, and I remain forever grateful to him for that. Through the windows he opened for me, I see his own works, those of Steve Reich, Iannis Xenakis, and Jean Claude Risset in developmental contexts. He brought me onto a path into New Music that I was subsequently able to continue on my own with the greatest pleasure.

The observations on perception psychology hinted at above are particularly central to the early compositions of Steve Reich, for example in *Piano Phase* (1967). I recently had the opportunity to discuss his early works with Steve Reich and was surprised that he did not perceive the links to current brain research and that these dimensions did not touch him either. And yet, one could also regard his compositions *Piano Phase* and *Come Out* (1966) as purely perceptual-psychological experiments. In Ligeti's case, the relationship between composition and effect is practically the opposite. He followed the development of brain research with great interest and with astonishing penetration, and there is no doubt that his enthusiasm for contemporary natural science was a unique motivation for his creation.

However, practically nothing is heard of this, and that is a good thing. Nevertheless, it is worthwhile to freely excavate the relationships and to look once more at *Continuum*.

In the right hand, we see a 2-pulse (B, G) and in the left hand also a 2-pulse (G, B). The synchronicity of both pulses or rhythms leads to a shimmering perception. After 9 "packages," which are marked by dashed vertical lines, Ligeti introduces a 3-pulse (B, G, F) in the right hand and maintains the 2-pulse (G, B) in the left hand. This tiny change now leads to a virtual rhythmic complexity in our hearing. A trace of chaos: Small differences can achieve a great effect. One could say, Ligeti built in a butterfly. Ligeti quote: "The harpsichordist must play very evenly and furiously fast there. But what we hear is not so much the individual notes, but tonal patterns that repeat at different frequencies - an acoustic illusion, in other words." Related compositional ideas are found again and again, e.g., in the first Etude *Désordre* (1985).

Now that we have touched upon a few traces of chaos in Ligeti, we want to make one more important point clear. Natural scientists like to speak of "deterministic chaos." With this, in addition to the definition of the term already mentioned above, they also want to make it clear that the character of unpredictability is to be distinguished from common randomness. One spoke of randomness before the discovery of chaos whenever an observation was made in which every attempt to find order failed. That is, making a prediction about the future course from the observed events is impossible. As an example, imagine someone flipping a coin and we observe the result. It is always heads or tails. But even after thousands of observations, we cannot predict whether the next toss will be heads or tails. We have the same problem at the roulette table. Here, too, there are no patterns for the sequence of results, at least if the table is in order. And yet, players keep trying to find patterns through observation and are inclined to believe that, for example, the number 7 must come up now if it hasn't in the last half hour.

Deterministic chaos means that one has a strict, even mathematical law according to which a natural process takes place, such as the weather on Earth, and yet the observations appear disorderly or like pure randomness, and small changes can have a great, surprising effect on the observer. We can practically equate mathematical law and algorithm, and when Ligeti is often placed in the context of algorithmic composing in connection with his works, it is worthwhile to cite a quote.

“Somewhere underneath, very deeply, there’s a common place in our spirit where the beauty of mathematics and the beauty of music meet. But they don’t meet on the level of algorithms or making music by calculation. It’s much lower, much deeper—or much higher, you could say.” Quote Ligeti from: Richard Steinitz. Music, Maths & Chaos. Musical Times, 137(1837):14–20, March 1996.

Nevertheless, Ligeti's conceptions, for example in *Désordre*, may be associated with the discovery of deterministic chaos. The common substrate lies in the deterministic nature of chaos, in contrast to the chaos, for example, in Xenakis, whose compositions were rather inspired by the traditional conception of randomness and stochastics.

Fractals: The Geometry of Chaos

Finally, we want to follow a last trace in Ligeti that leads us back to the beginning of our contribution. The discovery of deterministic chaos fascinated Ligeti as much as the fractal geometry of Benoît Mandelbrot. One image from my mathematical work particularly appealed to him, which is why I dedicated it to him in one of my books. Figure 2 shows this image – the Ligeti Fractal, a so-called Julia set, named after the French mathematician Gaston Julia, who conducted mathematical investigations at the beginning of the 20th century, which I continued with computer graphics means at the beginning of the 80s.

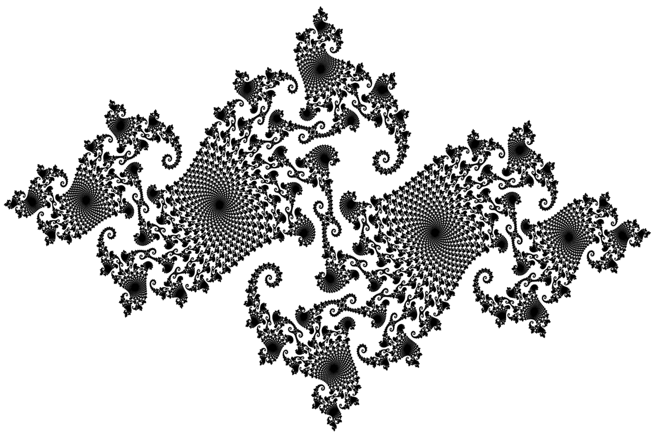

Figure 2. The Ligeti Fractal: Julia Set with the parameters a = (-0.745429,0.113008)

What are fractals now, and why did they impress and influence Ligeti so sustainably? Just as all objects made by humans have a shape that can be described with Euclidean geometry (line, square, rectangle, circle, cube, ...), nature has its own geometry. A cloud, a coastline in Croatia, an oak tree, or a vascular system in the liver, all have a geometry that cannot be grasped, described, or measured with Euclidean geometry. In the seventies of the 20th century, Mandelbrot created an extension of classical geometry with which the forms of nature can be described and measured just as well as human-made objects can be described by Euclidean geometry.

The Secret of Trees

Hardly anything seems more familiar to us than trees: deciduous trees, coniferous trees, vascular trees, ..., but also the trees in a hierarchical organization or the totality of all tributaries of a river, like the Elbe, or lightning in the sky come to mind. Botany defines trees as perennial, woody seed plants that have a dominant shoot that increases in circumference through secondary thickening growth. Mathematics also has an idea of trees: Trees are connected graphs in which any two nodes are connected by at most one path. Thus, all trees can be brought to a common denominator. It becomes more difficult when one has to describe how an oak structurally differs from a fir. It becomes even more difficult when one has to explain how our perception manages to distinguish an oak from a beech in a split second. And it is very difficult to understand how, for example, the portal vein as a tree in the liver can reach all liver cells and thus has a space-filling property.

For a long time, the relevant areas of mathematics (Geometry, Graph Theory, Topology, Statistics) could say little useful about this. This has fundamentally changed with fractal geometry. A pattern is called self-similar if the whole can be broken down into parts that look like small copies of the whole. If this property is only approximately given, one speaks of statistical self-similarity. The Sierpinski triangle in Figure 3 shows the exact self-similarity in a striking way. A fern frond compared to the whole fern is a beautiful example from nature.

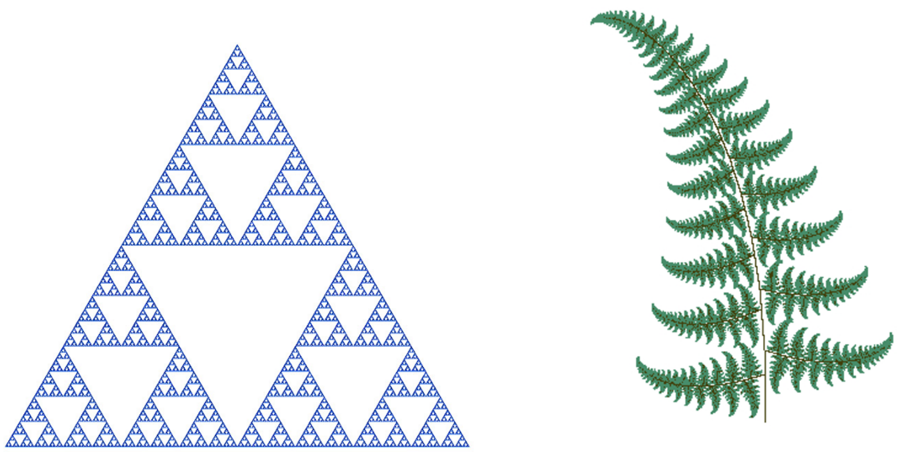

Figure 3. Strict self-similarity in the Sierpinski triangle and the Barnsley fern

If one now asks why a tree is the way it is, it becomes even more difficult. On the one hand, nature seems to play with variants, that is, to prefer a touch of randomness. On the other hand, trees are highly optimized organisms: For example, they organize geometric structures so that as many leaves as possible can be offered to the sun for photosynthesis at the same time. Only recently has it been understood that the geometry of trees also reduces the susceptibility to being knocked over by winds. And how does a tree grow? Is everything precisely encoded genetically in a seed and runs according to an unchangeable, predetermined growth program? We probably have to say goodbye to such a dominant interpretation of genetics. We recognize that many growth processes in animated and inanimate nature follow similar laws and mechanisms of self-organization, and self-similarity is typical for the patterns of self-organized processes.

New Elements of Geometry

Unlike Euclidean geometry, fractal geometry knows no elements like line, circle, ellipse, etc. In place of the elements, iterative or recursive processes enter fractal geometry. And here lies the connection to chaos. A particularly simple process, which at the same time was the pacesetter for the discovery of chaos, is the process:

$$\begin{cases} x_{n+1} = ax_n(1-x_n) \\ n = 0,1,2,3,.... \end{cases}$$

The number a is chosen fixed, e.g., $a = 2$. Then an initial number x_0 is chosen, e.g., $x_0 = 3$, and this value is inserted into the right side of the expression and one calculates. This completes the first iteration step and the result is inserted into the right side and one calculates. This completes the second step. In the third, one gets -195312. Now it is clear how it continues: the result of the last step is inserted and yields the next step. This is the meaning of the numbers n and n+1, which are appended to the numerical values x. Although there is hardly a simpler iterative process, it has been discovered that the process is chaotic for $a = 4$ and initial numbers x_0 chosen between 0 and 1 (i.e., e.g., $x_0 = 0.3356987...$). Today, this process is considered *the* prototype for chaos in general. Ed Lorenz was able to show in his groundbreaking work that the mechanisms for weather formation have something in common with this process. The process is even so chaotic that one could lose faith in computability by computers. For this, let's look at the following process in comparison:

$$\begin{cases} x_{n+1} = ax_n - ax_n^2 \\ n = 0,1,2,3,.... \end{cases}$$

One sees immediately that we have merely multiplied out the bracket in the first expression. The new process is thus mathematically identical to the old process. Only the order of the operations is changed. Let's take our example $x_0 = 3$ and now $a = 4$ to convince ourselves:

$$4 \bullet 3\ (1-3)$$

$$4 \bullet 3 - 4 \bullet 3^2$$

Now let's look at what a computer does with the calculations when we drive the iteration very far, e.g., by calculating 79 iterative steps and compare:

Process 1:

1	0.3	21	0.941784606	41	0.755660148	61	0.207807201
2	0.84	22	0.219305449	42	0.738551554	62	0.658493472
3	0.5376	23	0.684842276	43	0.772372624	63	0.899519277
4	0.99434496	24	0.863333332	44	0.703252615	64	0.36153739
5	0.022492242	25	0.47195556	45	0.834753498	65	0.923312422
6	0.087945365	26	0.996854038	46	0.551760383	66	0.283226373
7	0.32084391	27	0.012544262	47	0.989283451	67	0.812036779
8	0.871612381	28	0.049547612	48	0.042406818	68	0.610532194
9	0.447616953	29	0.188370585	49	0.162433919	69	0.951130536
10	0.989024066	30	0.611548432	50	0.544196564	70	0.185924957
11	0.043421853	31	0.950227789	51	0.992186655	71	0.605427469
12	0.166145584	32	0.189179751	52	0.031009187	72	0.955540195
13	0.554164917	33	0.61356309	53	0.12019047	73	0.169932523
14	0.988264647	34	0.948413698	54	0.422978883	74	0.564221843
15	0.046390537	35	0.195700621	55	0.97627099	75	0.98350222
16	0.176953821	36	0.629607551	56	0.092663775	76	0.064902415
17	0.582564664	37	0.932807531	57	0.336308798	77	0.242760365
18	0.972732305	38	0.250710565	58	0.892820762	78	0.735311082
19	0.10609667	39	0.751419111	59	0.382767396	79	0.77851478
20	0.379360667	40	0.747153723	60	0.945026066	80	0.68971807

Process 2:

1	0.3	21	0.941784606	41	0.755660148	61	0.207882282
2	0.84	22	0.219305449	42	0.738551554	62	0.658668955
3	0.5376	23	0.684842276	43	0.772372624	63	0.899296651
4	0.99434496	24	0.863333332	44	0.703252614	64	0.362248737
5	0.022492242	25	0.47195556	45	0.834753499	65	0.924098358
6	0.087945365	26	0.996854038	46	0.55176038	66	0.280562331
7	0.32084391	27	0.012544262	47	0.989283452	67	0.807388438
8	0.871612381	28	0.049547612	48	0.042406813	68	0.622049393
9	0.447616953	29	0.188370585	49	0.162433902	69	0.940415783
10	0.989024066	30	0.611548432	50	0.544196519	70	0.224135753
11	0.043421853	31	0.950227789	51	0.992186671	71	0.695595668
12	0.166145584	32	0.189179751	52	0.031009125	72	0.846969338
13	0.554164917	33	0.61356309	53	0.120190235	73	0.518449114
14	0.988264647	34	0.948413698	54	0.422978169	74	0.998638521
15	0.046390537	35	0.195700621	55	0.97627055	75	0.005438503
16	0.176953821	36	0.629607551	56	0.092665451	76	0.021635701
17	0.582564664	37	0.932807531	57	0.336314262	77	0.08467039
18	0.972732305	38	0.250710565	58	0.892827917	78	0.31000526

Both tables show 80 numerical values. The first one starts with 0.3 as the starting value and then calculates

$$4 \bullet 0.3\ (1-0.3) = 1.2 \bullet 0.7 = 0.84$$

The second table also begins with 0.3 as the starting value and then calculates

$$4 \bullet 0.3 - 4 \bullet 0.3^2 = 1.2 - 4 \bullet 0.09 = 1.2 - 0.36 = 0.84$$

Naturally, we get the same result in both cases. In both tables, the iteration is now further calculated with a computer, and we observe the same results step by step when comparing the two left columns, as we would naturally expect, because we are calculating with formulas that are mathematically equivalent. The second columns of the tables also match. But in the third column, we observe something unsettling: In position 44, the first process (i.e., after 43 steps) generates the value 0.703252615, while the second process offers the value 0.703252614 in the same position! Almost the same result, but not exactly the same result! In the 9th digit after the decimal point, we see a difference of 1, meaning the two results differ by 1 billionth. Practically the same result but not exactly the same result. And now the chaos becomes noticeable. This minuscule difference grows and grows when we compare positions 45 to 69. In position 69, the difference has already grown to the second decimal place, i.e., to approx. 1 hundredth. This corresponds to a growth factor of 10 million! And then it gets weirder and weirder. If we compare, for example, position 75, we see the value 0.98350222 in the first process and the value 0.005438503 in the second process! Considering that all results must always fall between 0 and 1, we have now reached a difference that is almost maximal. Which of the two processes is now correct and which is faulty? That cannot be determined. Chaos has struck. And how can we explain this? Well, one of the reasons lies in how computers calculate. They calculate quite accurately, but not entirely accurately. They cannot accept decimal numbers with very many digits. They can only accept decimal numbers with a few digits, say 16. All numbers that require more digits are radically rounded to 16. That sounds quite bad, but it is hardly noticeable for practical use, because the difference that arises from it is truly tiny. And chaos means that the smallest differences can grow into large consequences. But why do more and more decimal places actually come about through the iteration? Let's look at the first column: We start with one digit after the decimal point. In the first step, this becomes 2, then 4, then 8. Practically, the number of digits doubles in each step, and soon a tiny inaccuracy arises because the computer begins to round the digits. The two processes are mathematically equivalent and now calculate in a different order. In combination with the rounding, a tiny difference eventually arises, and chaos takes its course.

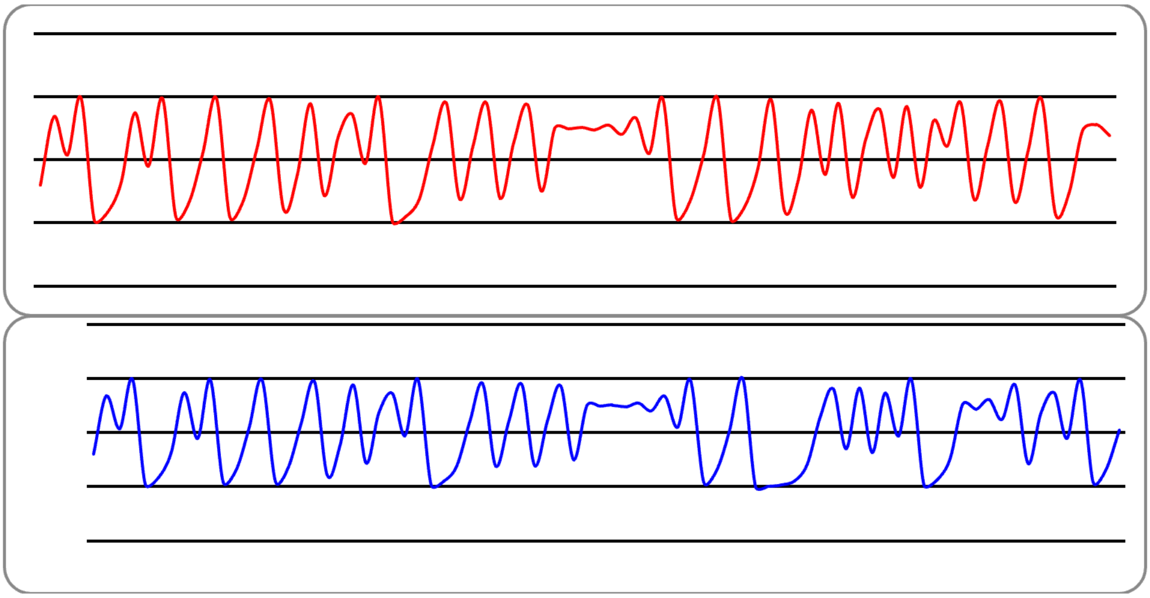

Figure 4. We represent Process 1 (top in red) and Process 2 (bottom in blue) as a temporal sequence. It is immediately noticeable that both processes appear in sync until a little over the middle and then deviate more and more from each other. An astonishing result, because in the computer calculation, the formula was merely replaced by the mathematically equivalent formula.

How difficult it is to see whether a process is chaotic or not becomes clear when one notes that the process is chaotic for $a = 4$ and for $a = 2$, for example, not at all. Then we can let both processes run side by side endlessly, and they always give the same answer. That is, the effect of rounding and the doubling of the digits alone explains nothing. It is not the cause of chaos. It only helps chaos get on its feet. Chaos lies much, much deeper in the process. If that were not the case, it would certainly have been discovered a long, long time ago.

If chaos can be hidden in such a simple calculation process, one suspects that this is not an isolated case. Indeed, iterative processes can easily slip into chaos. This is the case with the weather, for example: The weather of next week is calculated from today's weather with powerful computers and insanely intertwined calculation processes for a prognosis. How must one imagine this? Well, the computer starts with today's weather, say, at 12:00 noon. Instead of immediately calculating the weather for tomorrow and then the day after tomorrow, it proceeds and must proceed differently. It practically advances in second steps. That means the process is an iterative process with an insane number of steps until it arrives at tomorrow. If chaos prevails, the smallest inaccuracies, precisely the butterfly effect, can soon unfold a great effect.

One can imagine that Ligeti was fascinated beyond measure in our conversations when he unlocked these discoveries. But the matter of his fascination has not yet reached its endpoint. There is still something to add that belongs to the most beautiful discoveries of recent mathematics, and Ligeti, the curious one, also unlocked this in our conversations, and this culmination hints at a trace to the 4th movement of the Piano Concerto.

The process

$$\begin{cases} x_{n+1} = 4x_n(1-x_n) \\ n = 0,1,2,3,.... \end{cases}$$

can also be observed in other respects. For $x_0 = 3$, for example, the process, as we saw at the beginning, runs further and further into the negative. For x_0 between 0 and 1, however, it always yields values between 0 and 1. One can check that indeed for x_0 less than 0 and x_0 greater than 1, the values always plunge further into the negative, one could say, they *flee*. The set of starting values x_0 for which the iteration remains *trapped* is called the Julia set. If one now changes the choice of a to, for example, $a = 4.2$, there are new surprises. It remains the case that there are starting values x_0 for which the iteration remains trapped in one – the Julia set – and others for which the iteration plunges further and further into the negative. While the Julia set was the interval [0, 1] for $a = 4$, it is now, for $a = 4.2$, infinitely fragmented. Mathematicians speak of a Cantor set, a historical example of a fractal. The iteration remains trapped on this set and behaves chaotically, meaning the smallest errors can result in large deviations. As long as a is chosen greater than 4, this state of affairs always remains the same: the Julia set is fractal and the iteration on it is chaotic. This state of affairs may seem very special here and is the simplest example of one of the fundamental mathematical discoveries of the last part of the 20th century: Whenever a chaotic process is observed and the pattern the process draws is considered, a fractal is obtained. That is, fractals are the geometry of chaos, and self-similarity, or the repetition of the whole in the small, is a typical characteristic of chaos.

Let's look again at the example for $a = 4$. Here the Julia set was the interval [0, 1]. Is that also a fractal? Yes, but a degenerate fractal, but still a self-similar object. For one can divide the interval into two parts, e.g., [0, 1/2] and [1/2, 1], which look like the whole, only smaller.

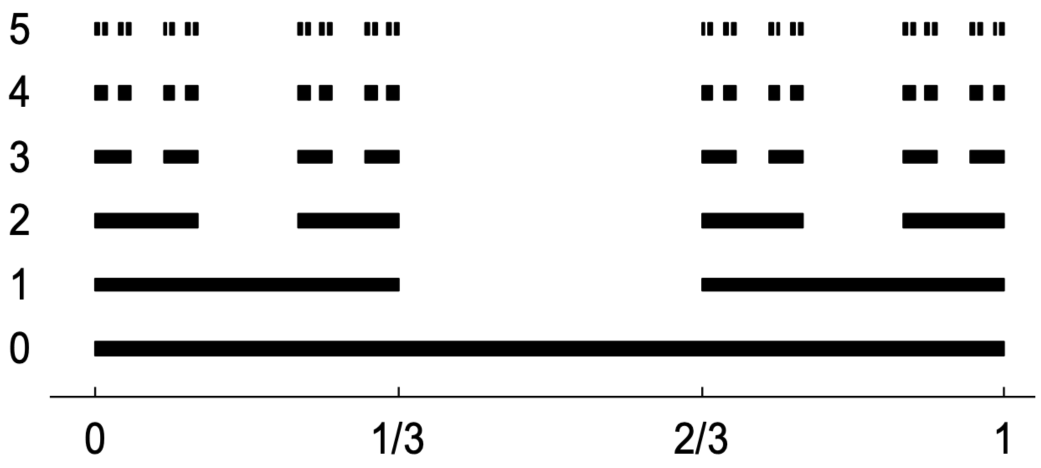

Figure 5. Step-by-step formation of a Cantor set: One starts with the interval [0, 1] as the initial set. In the first step, the interval is divided into three equal parts, and the middle interval (without its endpoints) is removed. The intervals [0, 1/3] and [2/3, 1] remain. In the second step, the middle intervals are again removed, and one now has the intervals [0, 1/9], [2/9, 1/3], [2/3, 7/9], and [7/9, 1]. In the following steps, one proceeds in the same way: the

middle parts are removed from the remaining intervals of the last step. In the 5th step, one already has $2^5 = 32$ parts, and if the iterative process is continued indefinitely, one has infinitely many parts in the end, the Cantor set. The iterative process describes a systematic fragmentation that ultimately leaves behind a dust.

We are soon at the culmination of our efforts, which are intended at least to indicate what fascinated Ligeti so much about Julia sets and especially about Figure 2. For this, we must consider the iteration

$$\begin{cases} x_{n+1} = ax_n(1-x_n) \\ n = 0,1,2,3,.... \end{cases}$$

once again from a new perspective. Instead of ordinary numbers, so-called complex numbers are now considered and iterated with them. Every point of a plane can be described as a pair z = (x, y) via a coordinate system. Calculating with complex numbers means nothing other than doing arithmetic with such pairs of numbers. The choice of *a* can then also be extended to such pairs of numbers.

$$\begin{cases} z_{n+1} = az_n(1-z_n) \\ n = 0,1,2,3,.... \end{cases}$$

Again, one has two behaviors for the iteration: Either the iteration runs further and further outwards in the plane – the iteration *flees*, or the iteration remains *trapped*, i.e., does not run further and further outwards. The Julia set now consists of all starting values z_0 for the iteration that remains trapped. Depending on how *a* is chosen, the Julia set is a dust, i.e., a Cantor set, or not. If it is a dust,

the iteration on the Julia set is chaotic, and this is how fractals and chaos come together. In recent decades, we have learned in mathematics and natural sciences that chaos and fractals form a pair: If a process is chaotic, it traces a fractal track, and often a chaotic process stands behind a fractal.

Ligeti worked through these dimensions in great depth in many conversations, and his fascination was comparable to the fascination of those who made the discoveries at the end of the 20th century.

If one looks at Bauhaus architecture, the concept stands naked before us and can be easily described by the elements of Euclidean geometry. The structure of a classical composition is also more or less simple to describe. In the case of a Ligeti Etude or "Poème Symphonique for 100 metronomes," this is hardly possible with the usual descriptive elements. Here, the process-like nature is formative, and this similarity to the structure formation in chaos is no accident with Ligeti. The Ligeti Fractal (Figure 2) had a special effect on him. This is not about aesthetics, but let us be clear that the constitutive process of this pattern is encoded by the simple formula, for . This is almost unimaginable and casts a new light on complexity in nature. We are accustomed to believing that a complex pattern can only arise through a complex process. Here we see a pattern that is unparalleled in its complexity while maintaining its harmony, and yet the process from which it arises could hardly be simpler. Ligeti also explored this again and again through probing questions and ventured into the most difficult mathematical cliffs. The fact that the eddies, as he called them, in the image I dedicated to him, inspired him to processes that play a major role in the 4th movement of the Piano Concerto should therefore not be understood superficially or even in the sense of program music.

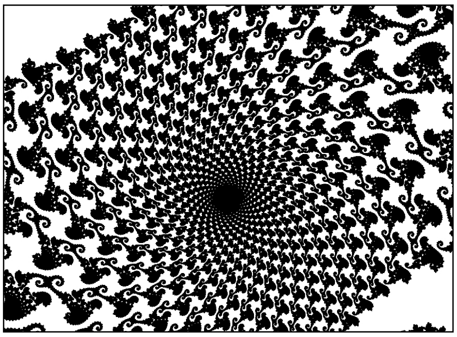

Figure 6. Excerpt from the Ligeti Fractal – the vortex.

I knew Ligeti as a persistent questioner who was anchored in music history as much as in his own time. One does not need to know that mathematics—especially chaos and fractals—and the sciences—especially the psychology of perception—fascinated him when succumbing to his music. But for him, both were more than just an inspiration. He felt like someone who participated in his work in his own way and made a contribution to these exciting fields. I think he succeeded uniquely.
When I think of Ligeti the *friend*, I think with great gratitude of the wonderful gift that binds me to him forever. On the occasion of a birthday, he called me and surprised me with the news that he wanted to dedicate the 17th Etude to me. But even small gifts are cherished. When we were on our way back to the hotel after a joint performance at the Music Festival in Huddersfield, I admired his magnificent umbrella. 'It's from Harrods in London and now it's yours,' he said briefly and succinctly. Any refusal to accept the gift was, of course, futile: 'If it makes you feel better, you can explain to me later in return how Mandelbrot actually came up with the concept of self-similarity.' That's how he was, my friend György Ligeti.